# Two-ness tasks

**Flipping cups**

You have N cups, all pointing upwards initially.

On any move, you can turn over any M of them.

Is it possible to have all N cups point downwards?

Quite obviously, if the problem is solvable for a given pair M, N then it is solvable for the pair qN, qM, with q a positive integer. For a long time I thought that the converse is also true, i.e. that the problems for (N, M) and the (reduced) pair M/***[gcd](http://www.cut-the-knot.org/blue/chinese.shtml#gcd)***(M, N), N/***[gcd](http://www.cut-the-knot.org/blue/chinese.shtml#gcd)***(M, N) are equivalent.

I was advised by the Zbarsky family that they are not. (For example, for N = 3 and M = 2 the problem has no solution. However, it is solvable for N = 6 and M = 4 in just three steps.)

When M and N are [***mutually prime***](http://www.cut-the-knot.org/arithmetic/Divisibility.shtml), the puzzle is solvable wherever M is odd, and unsolvable otherwise. Why?

It is easy to see that when N is odd and M is even, the puzzle is unsolvable. Indeed, assign number ±1 to each of the triangles depending whether it points up or down. Note the product Π of all the assigned numbers. When all triangles point upwards, Π = 1. When all point downwards, Π = -1. If M is even then flipping M triangles does not affect Π, and therefore, in this case, it is impossible to flip all the triangles.

If M = N - 1, there are C(N, N-1) = N [***combinations***](http://www.cut-the-knot.org/Generalization/cuttingcircle.shtml#comb) of N-1 elements out of N. Each of N elements enters N-1 of the combinations. Carrying out the flips corresponding to all N combinations, will turn each of the N triangles N-1 times, which is an odd number, and therefore leave it in a position different from the one it was originally in, i.e., upside down.

It remains to be shown that when gcd(M, N) = 1 and M odd the puzzle is always solvable. One solution can be found at [***my blog***](http://www.mathteacherctk.com/blog/2010/07/flipping-and-proving/).

It so happens that the case of an odd N is much easier than the case where N is even. For odd N, the problem can always be solved in 3 steps. When N is even, one needs at least 4 moves.

# Dual Problems

“Given three points in the plane, ﬁnd a fourth point such that the sum of its distances to the three given points is a minimum.” (Fermat, 17th Century)

“Given any triangle, circumscribe the largest possible equilateral triangle about it.”  
Annales de Math´ematiques Pures et Appliqu´ees”, edited by [J. D. Gergonne](http://en.wikipedia.org/wiki/Joseph_Diaz_Gergonne), vol. I (1810-11) p. 384